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On the Capacity of a Discrete, Constant Channel

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This paper is concerned with an iterative method for calculation of the capacity of a discrete, constant channel. Unlike the standard reference (Muroga 1953), in which Lagrange multipliers are used in the conventional form, the method described is based on techniques of mathematical programming. In addition, input signals of different duration are admitted. The algorithm proposed is very simple and has the advantage that it yields converging lower and upper bounds for the capacity.

1. STATEMENT OF THE PROBLEM

A discrete constant channel with m input and n output symbols is characterized by the $(n \times m)$ transition matrix

$$\left.\begin{array}{ll} P = (p_{ij}), & (i = 1, \cdots, n; j = 1, \cdots, m) \\ \text{with } p_{ij} \geqq 0, & \sum_i p_{ij} = 1. \end{array}\right\} \tag{1}$$

p_{ij} represents the conditional probability for receiving the i-th output symbol if the j-th input symbol has been transmitted. We assume that P contains no zero rows, i.e. no output symbols exist which will never be received.

In addition, a certain cost factor $t_j > 0$ is assigned to each input symbol, representing the cost (or time or energy) incident to the transmission of the j-th symbol.

For every input probability distribution $x = (x_j)$, the corresponding relative transmission rate $T(x)$ of the channel is defined as (Reza 1961)

$$T(x) = \left(\sum_j t_j\, x_j\right)^{-1} \sum_i \sum_j x_j\, p_{ij} \log \frac{p_{ij}}{\sum_k x_k\, p_{ik}}. \tag{2}$$

The relative capacity C of the channel is the maximum of T over all admissible input distributions x:

$$C = \max_{x \epsilon X} T(x) \tag{3}$$

with

$$X = \{x \mid x_j \geqq 0, \sum_j x_j = 1\}.$$

To simplify the notation, we introduce, in addition to t and x, the vectors α, y and $\log y$ with components

$$\alpha_j = \sum_i p_{ij} \log p_{ij}$$

$$y_i = \sum_j p_{ij} x_j ,$$

y_i being the output probabilities, and

$$(\log y)_i = \log y_i .$$

Further, we set

$$z = (x_j , y_i),$$

z being a vector with $m + n$ components, and

$$Z = \{z \mid x_j \geqq 0, \sum_j x_j = 1, y_i = \sum_j p_{ij} x_j\}.$$

The scalar product of two vectors is written as $\langle\ ,\ \rangle$. The transmission rate then becomes:

$$T(z) = \frac{\langle \alpha, x\rangle - \langle y, \log y\rangle}{\langle t, x\rangle} \equiv \frac{f(z)}{g(z)}, \tag{4}$$

and determination of the channel capacity C is equivalent to finding

$$C = \max_{z \in Z} T(z). \tag{5}$$

For P equal to the unit matrix (noiseless channel), the solution of this problem can be found for example in Reza (1961, p. 122). The case $t_j = 1$ for all j has been treated by Muroga (1953), Eisenberg (1963), and Bernholtz (1966).

The following properties can immediately be verified: Z is a compact convex polyhedron in Euclidean $(m + n)$-space. $f(z)$ and $g(z)$ are positive in Z. $T(z)$ is continuous in Z and continuously differentiable in the non-empty set

$$Z^0 = \{z \mid z \epsilon Z, y_i > 0 \text{ for all } i\}.$$

T assumes its maximum for at least one $\hat{z} \epsilon Z$.

It is well known (Feinstein 1958) that the function $f(z)$ is concave over Z. By definition, f is concave (or $-f$ is convex) over Z if the inequality

$$f(\theta z^1 + (1 - \theta)z^2) \geqq \theta f(z^1) + (1 - \theta)f(z^2) \tag{6}$$

holds for all $z^1, z^2 \epsilon Z$ and $0 < \theta < 1$. Further, a concave function f has the following support property:

$$f(z) \leqq f(z^0) + \langle \delta f(z^0), (z - z^0) \rangle, \tag{7}$$

where $\delta f(z^0)$ denotes the gradient of a function f at the point z^0. With respect to variations of the variables y only, f is even strictly concave, i.e., the equality sign is excluded in (6).

Due to the linear denominator, $T(z)$ is only a quasiconcave function over Z, which means that instead of (6), the weaker inequality

$$T(\theta z^1 + (1 - \theta)z^2) \geqq \min (T(z^1), T(z^2)) \tag{8}$$

holds for all $z^1, z^2 \epsilon Z$ and $0 \leqq \theta \leqq 1$. The inequality (8) is equivalent to the convexity of the level sets $C_u(T)$ (Mangasarian 1965):

$$C_u(T) = \{z \mid z \epsilon Z,\ T(z) \geqq u\}. \tag{9}$$

The latter condition is also true for concave functions, but it is not equivalent to (6). To prove that T has convex level sets, C_u has only to be written as

$$\begin{aligned} C_u(T) &= \{z \mid z \epsilon Z, f(z) - ug(z) \geqq 0\} \\ &= C_0(f - ug) \quad \text{for} \quad u > 0 \end{aligned}$$

and

$$C_u(T) = Z \quad \text{for} \quad u \leqq 0.$$

Since all the functions $f - ug$ are concave, the $C_u(T)$ are convex and T is shown to be quasiconcave.

The determination of the relative channel capacity can be viewed as a problem of quasiconcave programming. An iterative method for calculation of the capacity and the capacity achieving input distribution will be presented.

Remark. The method described is even applicable to more general cost functions. The following propositions 1 through 4 and theorems 1 and 2 remain valid for a denominator $g(z)$ having the following properties:

(1) $g(z)$ convex over Z
(2) $g(z)$ differentiable in Z with

$$g(z^1) + \langle \delta g(z^1), (z - z^1) \rangle > 0 \quad \text{for all} \quad z, z^1 \epsilon Z.$$

These are fulfilled for the denominator

$$g(z) = \langle t, x \rangle.$$

2. MATHEMATICAL PRELIMINARIES

Before describing our procedure, we need some simple propositions.

PROPOSITION 1. *If for two points $z^1, z^2 \epsilon Z$ and for a certain subscript i we have $y_i^1 = 0$, $y_i^2 > 0$, then z^1 cannot furnish a maximum of T on the line segment $[z^1, z^2]$.*

Proof. Due to the term $\langle y, \log y \rangle$ the one-sided directional difference quotient

$$\frac{T(z^1 + \lambda(z^2 - z^1)) - T(z^1)}{\lambda}, \qquad 0 < \lambda < 1,$$

tends to plus infinity as λ tends to zero.

PROPOSITION 2. *If $\hat{z}^1$ and $\hat{z}^2$ both furnish the maximum of T over Z then $\hat{y}^1 = \hat{y}^2$.*

Proof. Because of the quasiconcavity of T, every point of the line segment $[\hat{z}^1, \hat{z}^2]$ furnishes the same maximum, too. Therefore

$$f(z) = u \cdot g(z), \qquad u > 0, \qquad \text{for all} \quad z \epsilon [\hat{z}^1, \hat{z}^2].$$

However, a concave function is a positive multiple of a convex function if and only if both functions are linear. Therefore, y must not vary along the line segment $[\hat{z}^1, \hat{z}^2]$.

To formulate the next proposition, we introduce the function

$$\tau_{z^1}(z) = \frac{f(z^1) + \langle \delta f(z^1), (z - z^1) \rangle}{g(z^1) + \langle \delta g(z^1), (z - z^1) \rangle}. \tag{10}$$

Note that

$$\tau_{z^1}(z^1) = T(z^1)$$

and

$$\delta \tau_{z^1}(z^1) = \delta T(z^1).$$

PROPOSITION 3.

$$\tau_{z^0}(z) \geqq T(z) \quad \textit{for all} \quad z^0 \epsilon Z^0, z \epsilon Z.$$

Proof. Using the support property (7) for concave and convex functions, respectively, we increase the numerator and decrease the denominator to get

$$T(z) = \frac{f(z)}{g(z)} \leq \frac{f(z^0) + \langle \delta f(z^0), (z - z^0) \rangle}{g(z^0) + \langle \delta g(z^0), (z - z^0) \rangle} = \tau_{z^0}(z),$$

since the denominator is assumed to be strictly positive throughout.

Remark. It is clear from the argument used in the proof that the inequality even remains valid for a wider range of z^0, namely if all y_i^0 are larger than zero and $g(z^0) + \langle \delta g(z^0), (z - z^0) \rangle$ remains positive.

PROPOSITION 4. *If for $z \epsilon Z$, $z^0 \epsilon Z^0$ we have*

$$\tau_{z^0}(z) > \tau_{z^0}(z^0),$$

then

$$T(z^0 + \lambda(z - z^0)) > T(z^0)$$

for sufficiently small $\lambda > 0$.

Proof.

$$\frac{f(z^0) + \langle \delta f(z^0), (z - z^0) \rangle}{g(z^0) + \langle \delta g(z^0), (z - z^0) \rangle} > \frac{f(z^0)}{g(z^0)}$$

implies

$$g(z^0) \cdot \langle \delta f(z^0), (z - z^0) \rangle - f(z^0) \cdot \langle \delta g(z^0), (z - z^0) \rangle > 0,$$

which, divided by $[g(z^0)]^2$, gives $\langle \delta T(z^0), (z - z^0) \rangle > 0$. The proposition then follows from the last inequality and the continuity of the gradient.

We are now ready to establish the following theorem which characterizes the optimal solution of (5):

THEOREM 1. *$\hat{z}$ is optimal,* i.e. $\max_{z \epsilon Z} T(z) = T(\hat{z})$, *if and only if*

$$\hat{z} \epsilon Z^0 \quad \text{and} \quad \max_{z \epsilon Z} \tau_{\hat{z}}(z) = \tau_{\hat{z}}(\hat{z}).$$

Proof. $\hat{z} \epsilon Z^0$ follows from proposition 1. Proposition 3 shows the condition above to be sufficient, and the necessity follows from proposition 4.

3. THE PROCEDURE

Our iterative procedure is a modification of the Frank-Wolfe algorithm for quadratic programming (Berge and Ghouila-Houri 1962) and is composed of the following steps:

I. Start with $z^1 \epsilon Z^0$ arbitrarily.

II. Given $z^k \epsilon Z^0$, determine $\bar{z}^k \epsilon Z$ such that

$$\tau_{z^k}(\bar{z}^k) = \max_{z \epsilon Z} \tau_{z^k}(z).$$

III. Determine z^{k+1} on the line segment $[\bar{z}^k, z^k]$ such that

$$T(z^{k+1}) = \max_{z \epsilon [\bar{z}^k, z^k]} T(z).$$

Comments.

To I: Since P has no zero rows, z^1 with

$$x_j^1 = \frac{1}{m}, \qquad y_i^1 = \sum_j p_{ij} x_j^1,$$

is an element of Z^0 which can be chosen as a starting point.

To II: $\tau_{z^k}(z)$ is a fractional linear function which does not change its sign in Z. Replacing y by Px, we get

$$\begin{aligned} \tau_{z^k}(z) &= \frac{\langle \alpha, x \rangle - \langle y^k, \log y^k \rangle - \langle (y - y^k), (1 + \log y^k) \rangle}{\langle t, x \rangle} \\ &= \frac{\sum_j x_j [\alpha_j - \sum_i p_{ij} \log y_i^k]}{\langle t, x \rangle}. \end{aligned} \tag{11}$$

(11) holds for all y^k with $\sum_i y_i^k = 1$. Since the maximization of τ_{z^k} is a fractional linear programming problem, the maximum will be furnished by a vertex of X (Dinkelbach 1962). At such a vertex of X, one out of the x_j is equal to 1 and all other x_j vanish. Let us introduce the abbreviations

$$\sigma_j(y) = \frac{\alpha_j - \sum_i p_{ij} \log y_i}{t_j} \tag{12}$$

and

$$S(y) = \max_j \sigma_j(y). \tag{13}$$

To determine the vertex which furnishes the maximum of τ_{z^k}, a subscript j_k has to be chosen, for which

$$S(y^k) = \sigma_{j_k}(y^k).$$

Then $\bar{z}^k$ is given by

$$\bar{x}_{j_k}^k = 1, \qquad \bar{x}_j^k = 0 \quad \text{for} \quad j \neq j_k, \qquad \bar{y}_i^k = p_{ij_k},$$

and

$$\max_{z \in Z} \tau_{z^k}(z) = \tau_{z^k}(\bar{z}^k) = S(y^k).$$

To III: A function of one variable over a finite interval has to be maximized which can be done by standard methods. From proposition 4 follows the monotonicity of $T(z^k)$:

$$\text{either} \qquad T(z^{k+1}) > T(z^k)$$
$$\text{or} \qquad T(z^k) = \max_{z \in Z} T(z).$$

Proposition 1 shows that

$$z^{k+1} \in Z^0 \quad \text{if} \quad z^k \in Z^0.$$

We shall now prove:
THEOREM 2.

$$\lim_{k \to \infty} T(z^k) = \max_{z \in Z} T(z) \quad \textit{and} \quad \lim_{k \to \infty} y^k = \hat{y},$$

$\hat{y}$ being the unique y-part of each optimal point.

Proof. The sequence $\{z^k\}$, $z^k \in Z^0$, contains at least one point of accumulation $\hat{z} \in Z$. We can extract a subsequence $\{z^{k_\nu}\}$ of $\{z^k\}$ such that $\{z^{k_\nu}\}$ converges to $\hat{z}$ and the vertex determined in step II of the procedure is the same for all k_ν : $\bar{z}^{k_\nu} = \bar{z}$. (The latter is possible since we have only finitely many vertices, and at least one of them occurs infinitely often.) The convergence of the whole sequence $T(z^k)$ towards $T(\hat{z})$ is then assured by the monotonicity of the $T(z^k)$.
Let j_0 be such that $\bar{z}$ is described by

$$\bar{x}_{j_0} = 1, \qquad \bar{x}_j = 0 \quad \text{for} \quad j \neq j_0\,, \bar{y}_i = p_{ij_0}\,.$$

First, we want to prove $\hat{z} \in Z^0$. Otherwise a nonempty set I of subscripts i would exist with $\hat{y}_i = 0$ for $i \in I$ and $\hat{y}_i > 0$ for $i \notin I$.
Now, by the construction of $\bar{z}$,

$$\sigma_{j_0}(y^{k_\nu}) = \max_j \sigma_j(y^{k_\nu}) \quad \text{for all} \quad k_\nu\,. \tag{14}$$

Since p_{ij} contains no zero rows, for each $i \in I$ at least one of the terms $-p_{ij} \log y_i^{k_\nu}$ appearing in the expression for $\sigma_j(y^{k_\nu})$ tends to infinity as $y_i^{k_\nu}$ tends to zero. Thus the right-hand side of (14) tends to infinity, and consequently the left-hand side must also tend to infinity, which is only possible if

$$p_{ij_0} = \bar{y}_i > 0 \quad \text{for at least one} \quad i \in I,$$

i.e. there exists an i with

$$\hat{y}_i = 0 \quad \text{and} \quad \bar{y}_i > 0. \tag{15}$$

By the monotonicity of the $T(z^k)$ and by the construction in step III we have

$$\max_{z \in [z^{k_\nu}, \bar{z}]} T(z) = T(z^{k_\nu+1}) \leqq T(\hat{z}),$$

and therefore

$$\max_{z \in [\hat{z}, \bar{z}]} T(z) = T(\hat{z}). \tag{16}$$

From proposition 1 we conclude that (15) and (16) contradict each other; the set of subscripts I is empty and $\hat{z} \in Z^0$.

Because of theorem 1 it remains to be shown that

$$\tau_{\hat{z}}(\hat{z}) = \max_{z \in Z} \tau_{\hat{z}}(z).$$

From (16) and proposition 4 we have

$$\tau_{\hat{z}}(\bar{z}) \leqq \tau_{\hat{z}}(\hat{z}). \tag{17}$$

Further, the construction in step II gives

$$\tau_{z^{k_\nu}}(\bar{z}) = \max_{z \in Z} \tau_{z^{k_\nu}}(z),$$

and by continuity

$$\tau_{\hat{z}}(\bar{z}) = \max_{z \in Z} \tau_{\hat{z}}(z). \tag{18}$$

(17) and (18) show that even

$$\tau_{\hat{z}}(\bar{z}) = \tau_{\hat{z}}(\hat{z}). \tag{17$'$}$$

(17$'$) and (18) result then in the desired optimality condition:

$$\tau_{\hat{z}}(\hat{z}) = \max_{z \in Z} \tau_{\hat{z}}(z);$$

hence

$$C = T(\hat{z}) = \max_{z \in Z} T(z).$$

The convergence of the y-part y^k follows from the uniqueness of the optimal $\hat{y}$ according to proposition 2. This completes our proof.

Note that our iterations give lower and upper bounds for the capacity at each step. As already mentioned, the $T(z^k)$ converge monotonically from below to C.

On the other hand, from proposition 3 we conclude that

$$\tau_{z^k}(\bar{z}^k) \geqq C,$$

and from (17′) it follows that even

$$\lim_{\nu \to \infty} \tau_{z^{k_\nu}}(\bar{z}^{k_\nu}) = C.$$

Since the point of accumulation to which the sequence z^{k_ν} converges was arbitrarily selected, the same relation holds for every convergent subsequence and, therefore, for the whole sequence:

$$\lim_{k \to \infty} \tau_{z^k}(\bar{z}^k) = C.$$

In general, the $\tau_{z^k}(\bar{z}^k)$ do not converge monotonically.

In practical computations, the iterations will be stopped for a certain k_e as soon as the lower and upper bounds are equal within a prescribed accuracy. Then the y^{k_e} is an approximation to the optimal output distribution and the x^{k_e} approximates one possible capacity achieving input distribution. The sequence x^k of the input distributions itself does not necessarily converge. If the optimal output distribution $\hat{y}$ and the capacity C are given, the set of all input distributions achieving channel capacity is characterized by the following conditions:

$$\left.\begin{aligned} x_j &\geqq 0 \quad \text{for all } j \\ Px &= \hat{y} \\ \langle \alpha - C\cdot t, x\rangle &= -\langle \hat{y}, \log \hat{y}\rangle. \end{aligned}\right\} \tag{19}$$

The last condition is equivalent to the requirement that x maximize the linear form $\langle \alpha - C\cdot t, x\rangle$ under the first two conditions as constraints. This is a simple linear program and can be solved by well known standard methods (Vajda 1961).

It follows from the theory of linear programming that, if P has rank r, a solution $\hat{x}$ of (19) exists with at most r positive components (Vajda 1961). The theorem of Minty and Palermo (1963) that "an n-symbol receiver requires at most an n-symbol transmitter" is, therefore, also valid for our problem.

4. A RELATED MINIMUM PROBLEM

E. Eisenberg (1963) has shown for the special case $t_j = 1$ that the capacity can also be defined as the solution of a minimum problem. Since the paper of Eisenberg does not seem to be widely known and his proof is rather complicated, we want to derive this as an immediate consequence of our above results. The following discussion is only valid for the linear denominator.

Using the remark to proposition 3 and the comments to step II, we conclude from

$$T(z) \leqq \tau_{z^0}(z)$$

that

$$C \equiv \max_{z \epsilon Z} T(z) \leqq \max_{z \epsilon Z} \tau_{z^0}(z) \equiv S(y^0)$$

provided we have

$$y_i^0 > 0 \quad \text{for all } i \text{ and } \quad \textstyle\sum_i y_i^0 = 1.$$

On the other hand, for $z^0 = \hat{z}$, we get from theorem 1

$$C = \max_{z \epsilon Z} \tau_{\hat{z}}(z) = S(\hat{y}).$$

Thus C is also an extremum of the following minimum problem:

$$\left.\begin{aligned} &C = \min S(y) \\ &\text{subject to} \quad y_i > 0 \quad \text{for all } i \text{ and } \quad \textstyle\sum_i y_i = 1. \end{aligned}\right\} \tag{20}$$

Since $S(y)$ is monotonously decreasing in y_i, the constraint $\sum y_i = 1$ in (20) can be replaced by $\sum y_i \leqq 1$; the equality sign will still hold at the optimum point.

We have shown up to now that $\hat{y}$ is a solution of problem (20) provided $\hat{z}$ is a solution of (5). Since the objective function in (20) is strictly convex, the solution $\hat{y}$ of (20) is unique and is, therefore, equal to the y-part of the solution of (5). Thus the following duality theorem holds:

THEOREM 3 (Eisenberg 1963). *Both the following problems A and B have a solution and the extrema are equal:*

$$A\text{: Determine } \max_{z \epsilon Z} T(z)$$

$$B\text{: Determine } \min S(y)$$

$$\text{subject to } \textstyle\sum y_i \leqq 1, \qquad y_i > 0 \quad \text{for all } i.$$

If $\hat{z}$ is a solution of A, the y-part $\hat{y}$ is a solution of B. If $\hat{y}$ is a solution of B, then an $\hat{x}$ exists such that $\hat{z} = (\hat{x}, \hat{y})$ is a solution of A.

For problem B, Eisenberg (1963) has sketched a procedure which is only applicable to the case of two output signals where it yields the solution even in a finite number of steps. He conjectured that a similar finite procedure might exist for an arbitrary number of output symbols. We did not, however, follow this track but preferred to attack the problem in its original form.

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REFERENCES

BERGE, C., GHOUILA-HOURI, A. (1962), "Programmes, jeux et réseaux de transport." Dunod, Paris.

BERNHOLTZ, B. (1966), private communication.

DINKELBACH, W. (1962), Die Maximierung eines Quotienten zweier linearer Funktionen unter linearen Nebenbedingungen. *Z. Wahrscheinlichkeitstheorie* **1**, 141–145.

EISENBERG, E. (1963), On Channel Capacity, Internal Technical Memorandum M-35, Electronics Research Laboratory, University of California, Berkeley, California.

FEINSTEIN, A. (1958), "Foundations of Information Theory." McGraw-Hill, New York.

MANGASARIAN, O. L. (1965), Pseudo-Convex Functions. *J. SIAM Control* **3**, 281–290.

MINTY, G. J., PALERMO, R. V. (1963), An n-symbol receiver requires at most an n-symbol transmitter. *Trans. IEEE* IT-**9**, 123–124.

MUROGA, S. (1953), On the Capacity of a Discrete Channel. *J. Phys. Soc. Japan* **8**, 484–494.

REZA, F. M. (1961), "An Introduction to Information Theory." McGraw-Hill, New York.

VAJDA, S. (1961), "Mathematical Programming." Addison-Wesley, Reading, Massachusetts.